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A theoretical description of large viscoplastic shear deformation in metals

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Abstract In this work, a new 3-dimensional viscoplastic model based on a previous plasticity theory is presented. The proposed constitutive model anticipates the contribution of the main features of plastic behavior, such as yielding, rate effect, isotropic and kinematic hardening, through a new approximation of the constitutive equation with a viscoplastic term, as well as a new consideration of the functional form of the rate of plastic deformation. A high accuracy simulation of shear experimental data at various rates and temperatures for a variety of materials, as well as the sign inversion of normal stress has been postulated.

Introduction

Due to the fact that yield phenomenon is a main feature in soft metals, a large amount of theoretical work has been developed for the elastic–plastic behavior of polycrystalline solids. However, the corresponding constitutive equations for large deformations of these materials are still in a state of development. Among the various trends which undertake to formulate a complete set of constitutive equations for plastic behavior, many different approaches have been established. Some of these differences cause limited results on the related phenomena, but others are of important consequences for the whole formulation which occasionally is attempted. One of the main differences among the various contributions concerns the multiplicative decomposition of the deformation gradient tensor $\mathbf{F}$ [1]. According to this widely accepted assumption the tensor $\mathbf{F}$, which describes the way a material element $d\mathbf{X}$ deforms in a line element $d\mathbf{x}$in the present state, is separated into the elastic and plastic parts $\mathbf{F_e}$ and $\mathbf{F_p}$ correspondingly. These two tensors lack an explicit determination in the present configuration of the material elements, because each of them is referred to different configurational states. To avoid this problem a detailed description has been developed by Rubin [2, 3] who extended the ideas by Eckart [4] and Besseling [5]. In his work, an evolution equation has been specified including the relaxation effects of plastic deformation without introducing a plastic deformation tensor explicitly. This treatment can be made by introducing, at each material point, a vector triad $\mathbf{m_i}\,(i = 1, 2, 3)$, which models the orientation and elastic deformation of the crystalline regions relative to a reference lattice state, determined when the material is macroscopically stress-free. Since the vectors $\mathbf{m_i}$ characterize the atomic lattice in the present state, they are not directly connected to the material line elements $d\mathbf{X}$or $d\mathbf{x}$, but they can be used explicitly as a basis for tensors referred to the present configuration as well. Material anisotropies characterized by functions of these components are explicitly specified in the present state and consequently are trivially invariant under superposed rigid body motions. This formulation is in accordance with the ideas argued by both Besseling [5] and Mandel [6] that the rotation and deformation of the atomic lattice is not directly related to the total deformation, relative

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to the reference configuration of a continuum material element. Following this analysis, Rubin [2] avoids the introduction of separated spin tensors, and their constitutive relations, for expressing the corrotational rate of internal quantities necessary in describing various aspects of the plastic phenomena. These special entities called ''spins'' have extensively been used and subsequently formulated by Dafalias [7–10] in a different way than the common used material spin, which is related to the kinematics of deformation. In a recent work Dafalias [11] has systematically evaluated the concept of plastic spin based on classical hyper-elasticity, yield criteria and invariance requirements of the constitutive functions under superposed rigid body motion. The most decisive condition for the validity of a theory is the experimental evidences. Cho and Dafalias [12] have undertaken the concept of spin on the experimental data by Montheillet et al. [13]. In the present work an effort will be made to apply Rubin's [2] plasticity theory, on the same experimental data, and indirectly to compare the completeness of the different formulations. In doing so, at the beginning we will represent briefly the set of constitutive equations on a simple shear deformation made by Rubin [3], and going on we will solve the proper relations by introducing two modifications for a more suitable description, of the rate-dependent constitutive equations, and the functional form controlling the rate of plastic deformation.

Rubin's constitutive equations for simple shear large deformation

As mentioned in the introduction, the elastic deformation of each material point has been formulated through a triad of vectors $\mathbf{m_i}$, that are related to the dilatation, distortion and orientation of the mean atomic lattice in respect to some reference state. In the reference configurational state associated with the material, when it is stress-free, this triad of vectors constitutes a set of orthonormal vectors, implying that the corresponding metric tensor m_{ij} equal to $m_{ij} = \mathbf{m_i} \cdot \mathbf{m_j}$ is given by

$$m_{ij} = \delta_{ij} \tag{1}$$

In order to define the change of the volume element we are referred to, the dilatation J_m (which is unity in the reference lattice state) is introduced and given by

$$J_m = \mathbf{m_1} \times (\mathbf{m_2} \cdot \mathbf{m_3}) = (\det m_{ij})^{1/2} \tag{2}$$

Moreover, to define the distortional measures of the elementary volume, Rubin has introduced another set of orthonormal vectors $\mathbf{m}'_\mathbf{i}$ defined by the equations

$$\mathbf{m}'_\mathbf{i} = J_m^{-1/3}\mathbf{m_i} \text{ with } m'_{ij} = \mathbf{m}'_\mathbf{i}\mathbf{j}' = J_m^{-2/3} m_{ij} \tag{3}$$

It is easily then extracted that

$$\det m'_{ij} = 1 \tag{4}$$

The microstructural variable $\mathbf{m_i}$ are determined by an evolution equation of the form

$$\mathbf{m_i} = \mathbf{L_m}\mathbf{m_i} \tag{5}$$

where the second order tensor $\mathbf{L_m}$, corresponds to the elastic velocity gradient and is assumed to be separated additively into the form

$$\mathbf{L_m} = \mathbf{L} - \mathbf{L_p}, \quad \mathbf{L_p} = \mathbf{D_p} + \mathbf{W_p} \tag{6}$$

where $\mathbf{L}$ and $\mathbf{L_p}$, are the velocity gradients of total and plastic deformation, respectively, and $\mathbf{D_p}, \mathbf{W_p}$ are the symmetric and antisymmetric parts of the velocity gradients need to be specified by constitutive equations. It follows from (2), (3), (4) and (6) that

$$\frac{\dot{J}_m}{J_m} = \mathbf{D} \cdot \mathbf{I}, \quad \mathbf{D}' = -\frac{1}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{I}, \qquad \dot{m}'_{ij} = 2(\mathbf{D}' - \mathbf{D_p}) \cdot (\mathbf{m}'_\mathbf{i} \otimes \mathbf{m}'_\mathbf{j}) \tag{7}$$

where $\mathbf{D}'$ is the deviatoric part of the corresponding symmetric part $\mathbf{D}$ of the total velocity deformation gradient tensor $\mathbf{L}$. In extracting the above relations the plastic incompressibility

$$\mathbf{D_p} \cdot \mathbf{I} = 0 \tag{8}$$

has been used, while symbol $\otimes$ denotes the tensor product between two vectors,and $\mathbf{I}$ represents the identity tensor.

Concerning simple shear deformation in the $\mathbf{e}_1 - \mathbf{e}_2$ plane of a rectangular Cartesian base vector $\mathbf{e}_i$ with the initial values of the atomic lattice vectors $\mathbf{m}_i(0)$ to be equal $\mathbf{e}_i$, then it follows that $\mathbf{m_1}, \mathbf{m_2}$ will remain in the $\mathbf{e}_1 - \mathbf{e}_2$ plane while $\mathbf{m_3}$ will remain always parallel to $\mathbf{e_3}$. The velocity gradient tensor specified to describe such a deformation is given by the relation,

$$\mathbf{L} = \dot{\gamma}(\mathbf{e}_1 \otimes \mathbf{e}_2) \tag{9}$$

where γ(t) is a measure of shear strain. The symmetric and anti-symmetric part of this deformation will be given subsequently from the relations,

$$\mathbf{D} = \frac{\dot{\gamma}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + {}_2 \otimes \mathbf{e}_1), \quad \mathbf{W} = \frac{\dot{\gamma}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 - {}_2 \otimes \mathbf{e}_1) \tag{10}$$

The current values of vectors $\mathbf{m}_i$ may be expressed in terms of a rectangular Cartesian deformation tensor $\mathbf{F}_m$ relative to the initial values $\mathbf{m}_i(0)$ or the base vectors $\mathbf{e}_i$ of the reference lattice state as follows:

$$\mathbf{m}_i = \mathbf{F}_{mij}\mathbf{m}_j(0) = \mathbf{F}_{mij}\mathbf{e}_j \tag{11}$$

For the simple shear deformation with $\mathbf{m}_3$ always being parallel to $\mathbf{e}_3$ the following components of $\mathbf{F}_m$ will be 0,

$$\mathbf{F}_{m13} = \mathbf{F}_{m31} = \mathbf{F}_{m23} = \mathbf{F}_{m32} = 0 \tag{12}$$

Based on the evolution Eq. 5 for $\mathbf{m}_i$ and the additive decomposition Eq. 6 for $\mathbf{L}$ we obtain,

$$\begin{gathered} \mathbf{L}_m = \dot{\mathbf{F}}_m \cdot \mathbf{F}_m^{\mathrm{T}} \Rightarrow \dot{\mathbf{F}}_m = \mathbf{L}_m \cdot \mathbf{F}_m \\ \dot{\mathbf{F}}_m = (\mathbf{L} - \mathbf{L}_p) \cdot \mathbf{F}_m = (-\mathbf{D}_p - \mathbf{W}_p) \cdot \mathbf{F}_m \end{gathered} \tag{13}$$

consequently the time derivatives of the non-zero components of $\mathbf{F}_m$ will be given from the following equations:

$$\begin{gathered} \dot{\mathbf{F}}_{m11} = \mathbf{F}_{m11}(-\mathbf{D}_{p11} + \mathbf{F}_{m12}(\dot{\gamma} - \mathbf{D}_{p12} - \mathrm{W}_{p12}) \\ \dot{\mathbf{F}}_{m22} = \mathbf{F}_{m22}(-\mathbf{D}_{p12} + \mathrm{W}_{p12}) + \mathbf{F}_{m22}(-\mathbf{D}_{p22}) \\ \dot{\mathbf{F}}_{m33} = \mathbf{F}_{m33}(-\mathbf{D}_{p33}) \\ \dot{\mathbf{F}}_{m12} = \mathbf{F}_{m11}(-\mathbf{D}_{p12} + \mathrm{W}_{p12}) + \mathbf{F}_{m12}(-\mathbf{D}_{p22}) \\ \dot{\mathbf{F}}_{m211} = \mathbf{F}_{m21}(-\mathbf{D}_{p11}) + \mathbf{F}_{m22}(\dot{\gamma} - \mathbf{D}_{p12} - \mathrm{W}_{p12}) \end{gathered} \tag{14}$$

where D_{pij}, W_{pij} the rectangular Cartesian components of $\mathbf{D_p}, \mathbf{W_p}$ relative to $\mathbf{e}_i$ which may be verified from the corresponding flow rules.

Based on thermo-mechanical principles, the dissipation of the plastic work, and that materials have a plastically orthotropic behavior, Rubin [2] introduced the following flow rule for the symmetric part of the plastic velocity gradient tensor $\mathbf{D_p}$,

$$\mathbf{D_p} = \Gamma_\mathbf{p}\mathbf{D_p} \tag{15}$$

where $\Gamma_\mathbf{p}$ is a non-negative function expressing the rate of plastic deformation and needs to be specified, while $\mathbf{D_p}$ is its subsequent direction which for the simple shear case $(J_m = 1, \mathbf{m}'_i = \mathbf{m}_i)$ has the following form:

$$\begin{aligned} \mathbf{D_p} = & \frac{b_{11}}{2\mu}[\mathbf{T}' \cdot (\mathbf{m_1} \otimes \mathbf{m_1})](\mathbf{m_1} \otimes \mathbf{m_1} - \frac{1}{3}{}_{11}\mathbf{I}) \\ & + \frac{b_{22}}{2\mu}[\mathbf{T}' \cdot (\mathbf{m_2} \otimes \mathbf{m_2})](\mathbf{m_2} \otimes \mathbf{m_2} - \frac{1}{3}\mathbf{m}_{22}\mathbf{I}) \\ & + \frac{b_{33}}{2\mu}[\mathbf{T}' \cdot (\mathbf{m_3} \otimes \mathbf{m_3})](\mathbf{m_3} \otimes \mathbf{m_3} - \frac{1}{3}\mathbf{m}_{33}\mathbf{I}) \\ & + \frac{b_{12}}{2\mu}[\mathbf{T}' \cdot (\mathbf{m_1} \otimes \mathbf{m_2})](\mathbf{m_1} \otimes \mathbf{m_2} - \mathbf{m_2} \otimes \mathbf{m_1} - \frac{1}{3}\mathbf{m}_{12}\mathbf{I}) \\ & + \frac{b_{13}}{2\mu}[\mathbf{T}' \cdot (\mathbf{m_1} \otimes \mathbf{m_3})](\mathbf{m_1} \otimes \mathbf{m_3} - \mathbf{m_3} \otimes \mathbf{m_1} - \frac{1}{3}\mathbf{m}_{13}\mathbf{I}) \\ & + \frac{b_{23}}{2\mu}[\mathbf{T}' \cdot (\mathbf{m_2} \otimes \mathbf{m_3})](\mathbf{m_2} \otimes \mathbf{m_3} - \mathbf{m_3} \otimes \mathbf{m_2} - \frac{1}{3}\mathbf{m}_{23}\mathbf{I}) \end{aligned} \tag{16}$$

where the material constants b_{ij} $(i,j = 1,2,3)$ characterize the particular plastic response of the tested specimen. For isotropic plastic relaxation all values of b_{ij} are taken equal to unity, otherwise different values of these parameters should be endorsed to obtain anisotropic plastic behaviour. The description introduced in this work is going to be applied on such materials, which at small deformations respond isotropically, but at larger strains an orthotropic plastic response is observed. We have noticed that at small stretches the solution of the constitutive equations is irrelevant of the selection of b_{ij} values, leading to identical results. At the last chapter of this work we will apply a simulation for identical values of b_{ij} to verify this result. Taking this fact into consideration in our simulation procedure, we adopt different values for b_{ij} even at the initial deformations.

To complete the constitutive description, Rubin [2] has used the hyperelastic formula for the deviatoric part of the developed stress $\mathbf{T}'$, which in the case of simple shear deformation the corresponding relation has the form

$$\mathbf{T}' = G(\mathbf{m_r} \otimes \mathbf{m_r}) - \frac{1}{3}{}_{rr}I) \tag{17}$$

with G being the shear modulus.

After that, in order to construct a constitutive flow rule for the anti-symmetric part $\mathbf{W_p}$ of the velocity gradient tensor $\mathbf{L_p}$ Rubin based on the subsequent observation.

The time evolution of the directions $\mathbf{m}_i$ in the case where the material spin $\mathbf{W}$ vanishes is given from the equation

$$\dot{\mathbf{m}}_i = \mathbf{L}_m \cdot \mathbf{m}_i = (\mathbf{L} - \mathbf{L_p}) \cdot \mathbf{m}_i = (\mathbf{D} - \mathbf{D_p}) \cdot \mathbf{m}_i - \mathbf{W_p} \cdot \mathbf{m}_i \tag{18}$$

This formulations mean that the orientations $\mathbf{m}_i$ will remain constant if also $\mathbf{W_p}$ vanishes, and they will be aligned with the principal directions of $(\mathbf{D}-\mathbf{D_p})$ if $\mathbf{m}_i$ are orthogonal vectors. This remark will cause the triad $\mathbf{m}_i$ to rotate until $\mathbf{D}$ and $\mathbf{D_p}$ have the same principal directions. Supposing that this behavior will occur asymptotically even in the case where $\mathbf{W_p}$ does not vanish from the beginning of plastic deformation, in such a way Rubin specified a flow rule for the plastic spin as follows:

$$\begin{aligned}\mathbf{W_p} =&\omega_{12}[\mathbf{D_p}\cdot(\mathbf{m_1}\otimes\mathbf{m_2})]\cdot[\mathbf{m_1}\otimes\mathbf{m_2}-\mathbf{m_2}\otimes\mathbf{m_1}]\\ &+\omega_{13}[\mathbf{D_p}\cdot(\mathbf{m_1}\otimes\mathbf{m_3})]\cdot[\mathbf{m_1}\otimes\mathbf{m_3}-\mathbf{m_3}\otimes\mathbf{m_1}]\\ &+\omega_{23}[\mathbf{D_p}\cdot(\mathbf{m_2}\otimes\mathbf{m_3})]\cdot[\mathbf{m_2}\otimes\mathbf{m_3}-\mathbf{m_3}\otimes\mathbf{m_2}]\end{aligned} \tag{19}$$

where constants ω_{ij} control the rate at which this asymptotic transition takes place.

As Rubin proved in his work [3] in the case of elastic response, the elastic deformation tensor $\mathbf{B_m}$ and the deviatoric stress $\mathbf{T}'$ are independent of the original orientation of vectors $\mathbf{m_i}$ relative to the reference configuration. This leads to the conclusion that the introduction of vectors $\mathbf{m_i}$ can be fairly applied to isotropic materials in the elastic region. However, when plastic response emerges, the rotation of the triad vectors $\mathbf{m_i}$ is strongly dependent on their initial orientation, and their evolution also is influenced by the spin multiplier ω_{12}.

For such a description, the initial vectors $\mathbf{m_i}$ are given by the formulas

$$\begin{aligned}&\mathbf{m_1}=\cos\varphi_0\mathbf{e_1}+\sin\varphi_0\mathbf{e_2},\\ &\mathbf{m_2}=-\sin\varphi_0\mathbf{e_1}+\cos\varphi_0\mathbf{e_2},\quad {}_3=\mathbf{e_3}\end{aligned} \tag{20}$$

where ϕ_0 is the initial angle that vectors $\mathbf{m_i}$ form with vectors $\mathbf{e_i}$ of the reference direction.

In his relative work, Rubin simulated the plastic response in simple shear for various values of the pair ϕ_0 and ω_{12}. In his analysis, he has shown that the shear stress T_{12} is slightly influenced by these values, whereas the normal stresses T_{11} and T_{22} are strongly dependent on ϕ_0 and ω_{12}. Rubin also concluded that the evolution of angle ϕ is saturated in special directions, strongly influenced by ω_{12}.

Taking into account these observations, it seems inappropriate to apply this description in the case of an initially isotropic-plastically anisotropic material.

To overcome this inadequacy, in the simulation procedure all possible initial orientations of vectors $\mathbf{m_i}$ should be taken into account and the total stress could be obtained by summation of the result for each separate calculation. This means that the macroscopically isotropic material may be divided into various regions, each one with its own orientation. Assuming further that these regions deform to the same amount of strain, the overall stress will be the sum of each individual stress. To avoid this tedious procedure, an assumption has been made in the present work, where a specific initial value for angle ϕ_0, expressing a mean value of all possible orientations, combined with a specific value for spin coefficient ω_{12}, can be selected as material parameters to simulate the experimental results. The choice of these parameters has been made by trial and error in the simulation procedure, by demanding saturation values for the normal stresses up to large strains.

It is worthwhile to notice that in our simulation, shear stress T_{12}, as also in Rubin's original work, is actually not seriously affected by angle ϕ_0, in contrast to the normal stresses T_{11} and T_{22} which are strongly dependent on angle ϕ_0, exhibiting oscillation at high strains beyond yield. These oscillations disappear by a proper choice of parameters ϕ_0 and ω_{12}, certifying this way the validity of our assumption.

If the set of the above equations, constituted from the relations (14), the two flow rules (16), (19) for the quantities $\mathbf{D_p}$ and $\mathbf{W_p}$, and the constitutive Eq. 17 for the deviatoric stress tensor $\mathbf{T}'$, is accomplished with an expression for the functional form $\Gamma_\mathbf{p}$, will be obtained a complete plastic description for large simple shear deformation. This description can be applied in the case where material develops isotropic elasticity at the beginning, but it can emerge directional plastic response after that.

Ongoing this description however our contribution will be attentive on two modifications of the above presented analysis. The first one is related with the hyper-elastic constitutive equation of stresses in such a way accomplished with the rate dependence of yield phenomena at high temperatures. The second one is linked with a particular form of function $\Gamma_\mathbf{p}$, and supported with a physical assumption for mechanism which takes place inside materials under plastic deformation.

Three-dimensional, non-linear, rate dependence equation for stresses

In the case of material forming processes at high temperatures, the plastic behavior of large deformations is attended with intense non-linear time depended phenomena.

When uniaxial loaded is proceeded, the assumption of strain-hardening hypothesis employs the stress dependence of creep component with a power law form. The related description of such a process constitutes a helpful tool for obtaining a three dimensional rate dependence equation for stresses [14].

Taking into account the Cayley–Hamilton theorem applied to stresses, an exponential form of the deviatory stress tensor can be furnished as follows:

$$(\mathbf{T}'_{\mathbf{v}})^n = p_n(I_2, I_3)(_v')^2 + q_n(I_2, I_3)\mathbf{T}'_{\mathbf{v}} + r_n(I_2, I_3)\mathbf{I} \tag{21}$$

where p_n, q_n and r_n are polynomials in the invariants I_2, I_3 given by the relations,

$$I_2 = \frac{1}{2}T'_{ij}T'_{ij},_3 = \frac{1}{2}T'_{ij}T'_{jk}T'_{ki}, \quad \text{while} \quad I_1 = T'_{ii} = 0 \tag{22}$$

In the ensuing discourse we will attempt to extrapolate the uniaxial exponential form of stress dependence to the three dimensions. In this regard we employ the deviatory of the symmetric part of the velocity gradient tensor $\mathbf{D}'_m$ combined with the stress tensor $\mathbf{T}'_{\mathbf{v}}$ as follows:

$$\mathbf{D}'_{\mathbf{m}} = A(ap_n(I_2, I_3)(\mathbf{T}'_{\mathbf{v}})^2 + bq_n(I_2, I_3)\mathbf{T}'_{\mathbf{v}} + gr_n(I_2, I_3))\mathbf{I} \tag{23}$$

where A, a,b,g are combined constants. If we notice that $tr\mathbf{D}'_{\mathbf{m}} = 0$, and neglect the effect of the third invariant I_3 for all values of n, then polynomials p_n are vanishing, given that they are proportional to I_3 as can be proved with recursive calculation of exponential forms of deviatory tensor. Taking this observation into account we can assume a simple form of dependence from the invariant I_2 via the relation $q_n = I_2^n$, where n is a non-negative integer. Considering these simplifications we obtain a simple form for the symmetric part of the velocity gradient under constant stress as follows:

$$\mathbf{D}'_m = AI_2^n\mathbf{T}'_{\mathbf{v}} = 0, 1, 2, 3, \ldots \tag{24}$$

Solving Eq. 23 for $\mathbf{T}'_{\mathbf{v}}$, we immediately obtain

$$\mathbf{T}'_{\mathbf{v}} = \frac{\mathbf{D}'_{\mathbf{m}}}{AI_2^n}, \quad I_2 = \frac{1}{2}\mathbf{T}'_{\mathbf{v}} \cdot \mathbf{T}'_{\mathbf{v}} = \frac{'_m \cdot D'_m}{2A^2I_2^{2n}} \tag{25}$$

The inversion is not complete until we express on the right side of Eq. 25a in terms of a function of the tensor $\mathbf{D}'_m$. Thus we use these equations to define the following expression:

$$K_2 = \frac{1}{2}\mathbf{D}'_{\mathbf{m}} \cdot \mathbf{D}'_{\mathbf{m}} \tag{26}$$

which will be recognized as the second invariant of the deviator tensor $\mathbf{D}'_m$. Now going back to Eq. 25 and introducing K_2 as given above, we obtain after solving for I_2

$$I_2 = A^{-2/2n+1}K_2^{1/2n+1} \tag{27}$$

Finally, eliminating I_2 from Eq. 25 by means of Eq. 27 we obtain after simplification

$$\mathbf{T}'_{\mathbf{v}} = A^{-1/2n+1}K_2^{-n/2n+1}\mathbf{D}'_{\mathbf{m}} \tag{28}$$

We have thus obtained a useful inverted form of a three-dimensional stress component depended from the deformation rate. Assuming that this viscous stress is developed in parallel with an hyperelastic stress component given by Eq. 17, a non-linear dependence from the deformational rate of the total stress will be resulted

$$\mathbf{T}'_{\text{total}} = G(\mathbf{m_r} \otimes \mathbf{m_r} - \frac{1}{3}\mathbf{m_{rr}I}) + A^{-1/2n+1}K_2^{-n/2n+1}\mathbf{D}'_{\mathbf{m}} \tag{29}$$

This expression can be easily applied in the case where $\mathbf{D}'_m$ is not varied in respect of time during deformation, however in our case the evolution equations for the triad of vectors $\mathbf{m_i}$ result to a strong time dependence of the velocity gradient tensor at initial deformation becoming almost constant at large stretches, so Eq. 29 could only be applied to describe the rate effect at large deformations. To overcome this deficiency, an alternative description will be adopted. In the sequel we will assume a linear dependence of viscous stress from deformational rates. In this case a superposition principle can be established. Using a relaxation function $\Psi(t) = C \exp(-t/t_e)$ with a relaxation time constant t_e, we can construct a corrotational integral of Jaumann type for the viscous stress $\mathbf{T}'_{\mathbf{v}}$ as follows:

$$\mathbf{T}'_{\mathbf{v}} = \int_0^t \Psi(t - t')\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'} \cdot \mathbf{D}'_m \cdot {}^{t'}_{t}\, dt' \tag{30}$$

where $\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}$ is the material matrizant as has been defined by Coddard and Chester [15] to obtain a three dimensional rheologigal model satisfying the objectivity of deviatoric stress tensor $\mathbf{T}'_{\mathbf{v}}$ under a superposed rigid body rotation.

If the anti-symmetric part $\mathbf{W_m}$ of the velocity gradient tensor $\mathbf{L_m}$ is taken into account as the rotational rate of the superposed rigid body motion, the definition of material matrizant is given according to Ref [15] by the form,

$$\frac{D\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}}{Dt} = \mathbf{W_m} \cdot \mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}, \quad \frac{\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}}{Dt} = -\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'} \cdot \mathbf{W_m} \tag{31}$$

with

$$\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}''} = \mathbf{I} \quad \text{whenever} = t'.$$

Since $\mathbf{W_m}$ is anti-symmetric the material matrizant satisfies the property [15],

$$[\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}]^{\mathrm{T}} = [^{t}_{t'}]^{-1} = [\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}''}]$$

Hence the tensor $\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}$ can be interpreted as an orthogonal transformation.

Following Eq. 14, tensors $\mathbf{D}'_m, \mathbf{W}_m$ can be numerically calculated for each time step, so the material matrizant tensor $\mathbf{Q}^{\mathbf{t}}_{\mathbf{t}'}$ can also be obtained numerically at every integration step using Eq. 31. By matching now the results of Eqs. 17 and 30 the total deviatoric stress is given from the expression,

$$\mathbf{T}'_{\text{total}} = G(\mathbf{m_r} \otimes \mathbf{m_r} - \mathbf{m}_{rr}\mathbf{I}/3) + \int_0^{\mathrm{t}} \Psi(t-t')^{\mathrm{t}}_{\mathrm{t}'} \cdot \mathbf{D}'_m \cdot \mathbf{Q}^{\mathbf{t}'}_{\mathbf{t}} \, \mathrm{d}t' \tag{32}$$

The functional form controlling the rate of plastic deformation

Following Rubin's analysis for the way in which plastic relaxation phenomena are described in respect to reference configuration, we remind the abundance of intermediate configuration necessary for the polar decomposition of the deformation gradient tensor **F**. In this contain, we are confident to suppose that plastic deformations coexist and accumulated moderately with the corresponding elastic ones, before they emerged macroscopically. This accumulation is taken part around a large number of anomalies randomly distributed into the volume of the deformed material. At this stage we will make use of the concept of distributed strain energy or equivalently of strain distribution around each anomaly responsible for plastic emergences. This idea has been applied successfully in amorphous materials as thermoplastic polymers [16, 17]. In amorphous materials the origin of random strain distribution into the deformed volume element was the concept of free volume, which as is known very well from polymer behavior, constitutes the seed for plastic emergence into the bulk under deformation. What is novel in the case of polycrystalline metals is the fact that these disturbances are usually constituted from the various kinds of dislocations emerged inside the crystal. The total applied deformation will consequently be distributed inhomogeneously around each separate dislocation, in a way related with both, the special features of each anomaly, and the relative orientations of slip planes that are associated with the directions of applied stresses.

When the distributed elastic energy around each dislocation reaches a critical value, a non-reversible transition takes place, which manages the emergence of plastic deformations. If we accept that each one of these transitions proceeds with a constant rate, then the macroscopic plastic deformations will come out with a rate proportional to the number of simultaneously appeared transformations. Ongoing we suppose that the necessary strain which is accumulated around the *i*-dislocation randomly selected from the statistical ensemble, obeys a normal Gaussian distribution [16, 17] determined from a mean equivalent strain $\tilde{\mu}$, and a standard deviation $\tilde{s}$. Then the distribution density function in respect to equivalent strain $\tilde{e}_i$ as a random variable will be given by

$$f(\tilde{e}_i) = \frac{1}{\tilde{s}\sqrt{2\pi}} \left[-\frac{1}{2} \left(\frac{\tilde{e}_i - \tilde{\mu}}{\tilde{s}} \right)^2 \right] \tag{33}$$

The application of an equivalent strain field $\tilde{e}$, applied with an equivalent strain rate $\dot{\tilde{e}}$, activates the process of nucleation, growth and merging of plastic transformations. The fraction of such processes that have enough activation energy to attain a new non-reversible state is given by the probability

$$P(0 < \tilde{e}_i < \tilde{e}) = F(\tilde{e}) - F(0) = \frac{1}{\tilde{s}\sqrt{2\pi}} \int_0^{\tilde{e}} \mathbf{exp} \left[-\frac{1}{2} \left(\frac{\tilde{e}_i - \tilde{\mu}}{\tilde{s}} \right) \right] \mathrm{d}\tilde{e}_i \tag{34}$$

Making the further assumption that the rate of plastic deformation $\Gamma_\mathbf{p}$ is proportional to the fraction of plastic transformations that have achieved a non-reversible state, and this transition takes place with an average rate $\dot{k}$ for every plastic transformation then we have

$$\Gamma_p = \dot{k} P(0 < \tilde{e}_i < \tilde{e}) = \dot{k} \frac{1}{\tilde{s}\sqrt{2\pi}} \int_0^{\tilde{e}} \mathbf{exp} \left[-\frac{1}{2} \left(\frac{\tilde{e}_i - \tilde{\mu}}{\tilde{s}} \right)^2 \right] \mathrm{d}\tilde{e}_i \tag{35}$$

The value of $\dot{k}$ can be estimated, assuming that at the yield point,that is, at the moment where the equivalent strain $\tilde{e}$, is equal to the mean value $\tilde{\mu}$, the rate of plastic deformation $\Gamma_{\mathbf{p}}^{\mathbf{y}}$ will become equal to the applied equivalent strain rate $\dot{\tilde{e}}$

$$\Gamma_{\mathbf{p}}^{\mathbf{y}} = \dot{\tilde{e}} = \dot{k}\frac{1}{\tilde{s}\sqrt{2\pi}}\int_0^{\tilde{\mu}} \mathbf{exp}\left[-\frac{1}{2}\left(\frac{\tilde{e}_i - \tilde{\mu}}{\tilde{s}}\right)^2\right]\mathrm{d}\tilde{e}_i = \dot{k}\frac{1}{2} \tag{36}$$

then

$$\Gamma_{\mathbf{p}} = \frac{2\dot{\tilde{e}}}{\tilde{s}\sqrt{2\pi}}\int_0^{\tilde{e}} \mathbf{exp}\left[-\frac{1}{2}\left(\frac{\tilde{e}_i - \tilde{\mu}}{\tilde{s}}\right)^2\right]\mathrm{d}\tilde{e}_i \tag{37}$$

In this expression we can endorse now the most common type of hardening which usually accompanies plastic deformation of metals, i.e. the isotropic and kinematic hardening as well. As far as isotropic hardening conveys, the value of parameter $\tilde{s}$ is adequate to describe the corresponding effect, given that it is strongly related with the broad or narrow distribution of function (36). Consequently we can state, that whatever an intense isotropic hardening emerges during plastic deformation, a broad distribution function should be endorsed calibrated from a corresponding value of $\tilde{s}$. On the other hand, kinematic hardening will be controlled by the evolution of the equivalent mean value of strain $\tilde{\mu}$. When plastic deformation proceeds and the development of some dislocations attached to a kind of some unsurpassed border, a larger value of $\tilde{\mu}$, should be endorsed to describe the new state of yield surface for the evolution of plastic deformation. A simple rate equation describing this evolution starting from an initial value $\tilde{\mu}_0$, and saturated at a value $\tilde{\mu}_s$ can be written as follows:

$$\dot{\tilde{\mu}} = \alpha(\tilde{\mu}_s - \tilde{\mu}) \tag{38}$$

where α is a constant controlling the rate of kinematic hardening. In the case of cycling loading, different values for constants $\tilde{\mu}_s, \tilde{\mu}_0$ should be adopted under reversed loading to describe the Bauschinger effect.

Experimental verification and discussion

The set of constitutive equations presented above has to be tested by experimental evidences, and compared with previous theoretical works. The special application of this theoretical model on the plastic behavior of materials under simple shear deformation, constitutes a decisive examination of the made assumptions. Such a trial is due to the fact that simple shear tests exhibit valuably most of the related phenomena accompanying the plastic behavior. Such phenomena concern the anisotropic plastic response, orientational hardening, rate and temperature dependence, etc. Special interesting however displays the sign variation of normal stresses before steady state is saturated under large deformations. The widely used experimental tests for obtaining such deformational conditions are torsion tests because large strains can be readily imposed owing the necessary geometrical and mechanical stability. Such type of experiments had been executed repeatedly from many researches [18] in a lots of metals and alloys, where interesting phenomena of the plastic behavior have been recorded at various temperatures. For the requirements of this article however we will be concentrated on the work of Montheillet [13] where systematic torsion tests have been executed on various materials for a wide range of temperatures and deformational rates. Results of such experiments have been used from many researchers as representative examples for the verification of their theoretical approaches. Dafalias and Cho [12] have tested systematically the assumption and the special task of the concept of plastic spin under large viscoplastic deformation, which has been introduced and theoretically founded by Dafalias in previous works [10, 11]. The target of this article is to test and compare the theoretical description of plastic behavior introduced by Rubin [2] with the proper comprehensiveness presented above, and consequently we will present the same sequence of experimental results contained in the work of Montheillet as used by Dafalias and Cho [12].

The first approach concerns the description of plastic behavior of Al under 200 °C, where torsion tests have been recorded an axial force corresponding to a normal stress of a representative element under shear deformation in three different strain rates imposed.

Figure 1a presents the response of compressive normal stress measured for Al at 200 °C for three different strain rates. The corresponding curves are incorporated in terms of true stress–strain results by Dafalias and Cho [12] after the original experimental work by Montheillet et al. [13]. Figure 1b shows the corresponding theoretical plots as have been calculated from the set of equations presented above and Fig. 1c presents the same plots obtained by the solution of the constitutional equations based on the concept of plastic spin [12]. The set of the corresponding differential

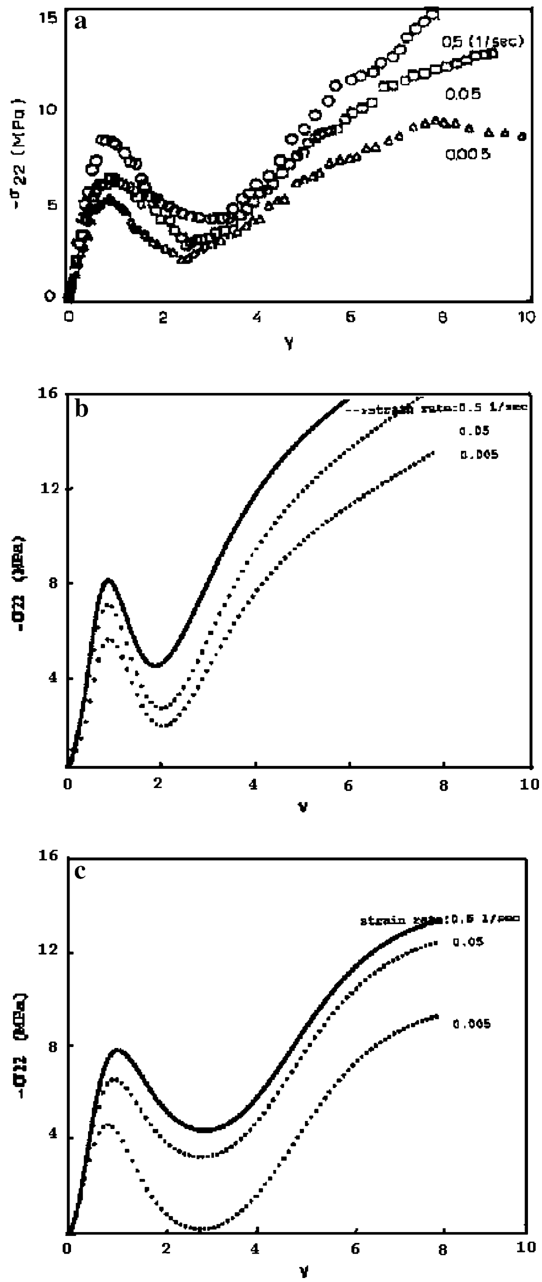

Fig. 1 (**a**) Experimental axial stress–strain curves for Al at 200 °C and different strain rates after Montheillet et al. [13]. (**b**) Simulation of the experimental results for Al at 200 °C, with the proposed model. (**c**) Simulation of the experimental results for Al at 200 °C, after Cho and Dafalias [12]

equations are solved numerically by programming using the soft work "Mathematica-4" by Wolfram [19]. The results shown in the c parts of these figures are obtained by solving the corresponding equations of Ref [12] by the same numerical method as in our set of equation has been done. By this process we have assure the same accuracy of the applied method, on the other hand the results obtained by our approach have been compared with the corresponding plots of relative reference [12]. The inclusive coincidence, that is observed from this comparative study, verifies the validity of our approach.

Table 1 Model constants used for the simulation of experimental results (Figs. 1–3)

Model constants	Al (200 °C)	Cu (300 °C)	α-Fe (800°C)
G (MPa)	25	27.5	28
$\tilde{\mu}$	0.5	1.0	0.1
$\tilde{s} = \tilde{\mu}/3$	0.5/3	1.0/3	0.1/3
$b_{11} = b_{22}$	0.4	0.7	0.6
$b_{12} = b_{13} = b_{23} = b_{33}$	1	1	1
ϕ_0 (rad)	0.863	0	0
ω_{12}	3	–6.5	–8
$\omega_{13} = \omega_{23}$	0	0	0
C (MPa)	5	0.5	0.1
t_e (s)	100	1	5

The values of material constants used for the calculation of these results are contained in Table 1. For most of the simulations, plastic relaxation is taken to be anisotropic in the sense that the values of b_{ij} are not equal to unity. The determination of these parameters has been based on the different influences which take part by matching them in three separated groups. The first group $(G, \tilde{\mu}, \tilde{s})$ is related with original features of plastic behavior, yield stress, saturation values of σ_{12}, isotropic and kinematic hardening. The second group (b_{ij},ω_{12}) is monitored from the orientational hardening and the influence of the oscillatory behavior of normal stresses. The third group of parameters (C, t_e) determines the effect of strain rate and is obtained by simulating the relative position of saturated values at large strains for shear and normal stresses as well.

Figure 2a–c, show the corresponding experimental and theoretical results of normal stresses for Cu tested at 300 °C for three different rates of deformation. Material parameters are also contained in Table 1. The theoretical results of Fig.2c, that are obtained by the solution of the plastic spin theory, by Dafalias and Cho [12], exhibit an inversion in the saturated values of normal stresses for strain rates 0.5 and 0.05 s^{-1} correspondingly, that is not obtained in the experimental data or the theoretical predictions of Fig. 2b.

Figure 3a, is drawn from the torsion tests of α-Fe at elevated temperature of 800 °C. In this figure, shear stress is plotted for a strain rate 0.5 s^{-1} and normal stresses for three different strain rates, 0.007, 0.05, 0.5 s^{-1}. The corresponding theoretical results are shown in Fig. 3b and c. The theoretical curve of shear stress of Fig. 3b is very close to the corresponding experimental, while the theoretical prediction of Fig. 3c exhibits a deviation of the order of 20%.

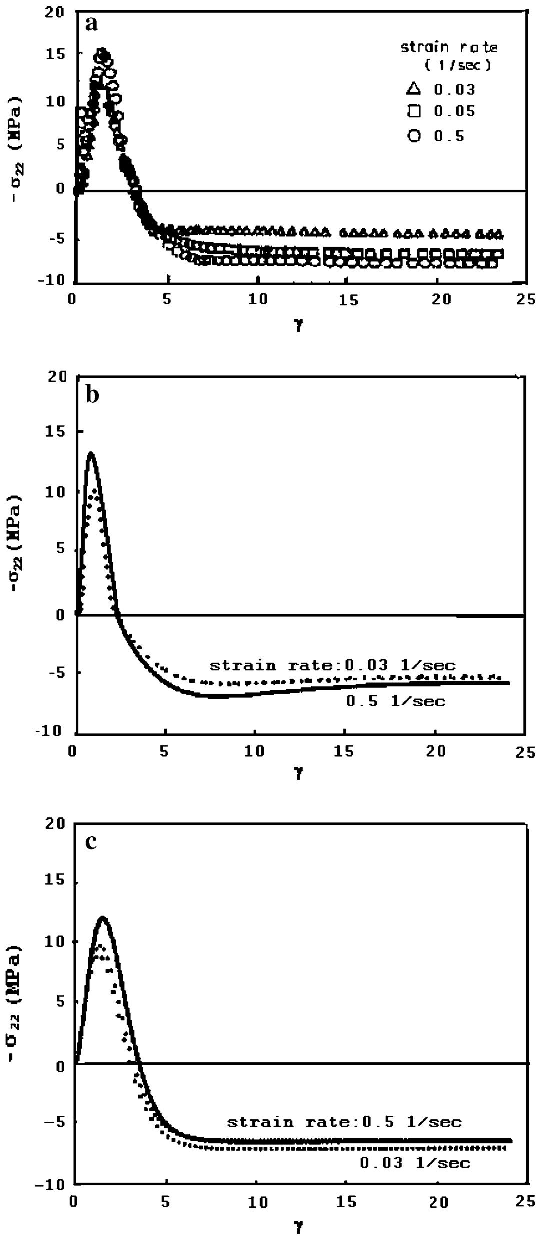

Fig. 2 (**a**) Experimental axial stress–strain curves for Cu at 300 °C and different strain rates after Montheillet et al. [13]. (**b**) Simulation of the experimental results for Cu at 300 °C, with the proposed model; (**c**) Simulation of the experimental results for Cu at 300 °C, after Cho and Dafalias [12]

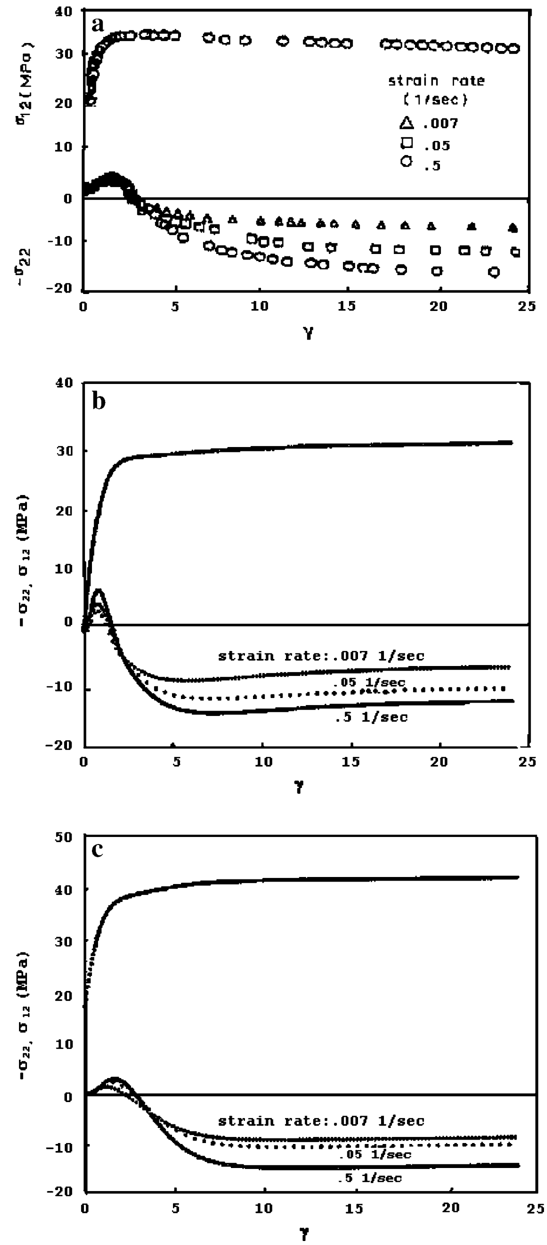

Fig. 3 (**a**) Experimental stress–strain curves for α-Fe at 800 °C and different strain rates after Montheillet et al. [13]. (**b**) Simulation of the experimental results for α-Fe at 800 °C, with the proposed model. (**c**) Simulation of the experimental results for α-Fe at 800 °C, after Cho and Dafalias [12]

Figure 4a–c are drawn for the experimental results and theoretical approximations of normal stresses at various temperatures for Al under a certain strain rate deformation of 1/s. The values of the material parameters used at different temperatures are included in Table 2. As can be noticed from this table, simulating the temperature dependence, among the parameters which have to be changed, are the shear modulus and constant ω_{12} which controls the rotation rate and the role of material spin in aligning the triad of vectors $\mathbf{m_i}$ with the principal directions of **D**.

Figure 5a and b reproduces the simulations obtained at the beginning of this section with the proposed model for axial stress–strain curves in Al and Cu, included at 1a and 2a plots. These results however are compared with simulation obtained for isotropic response where all values of b_{ij} are taken equal to

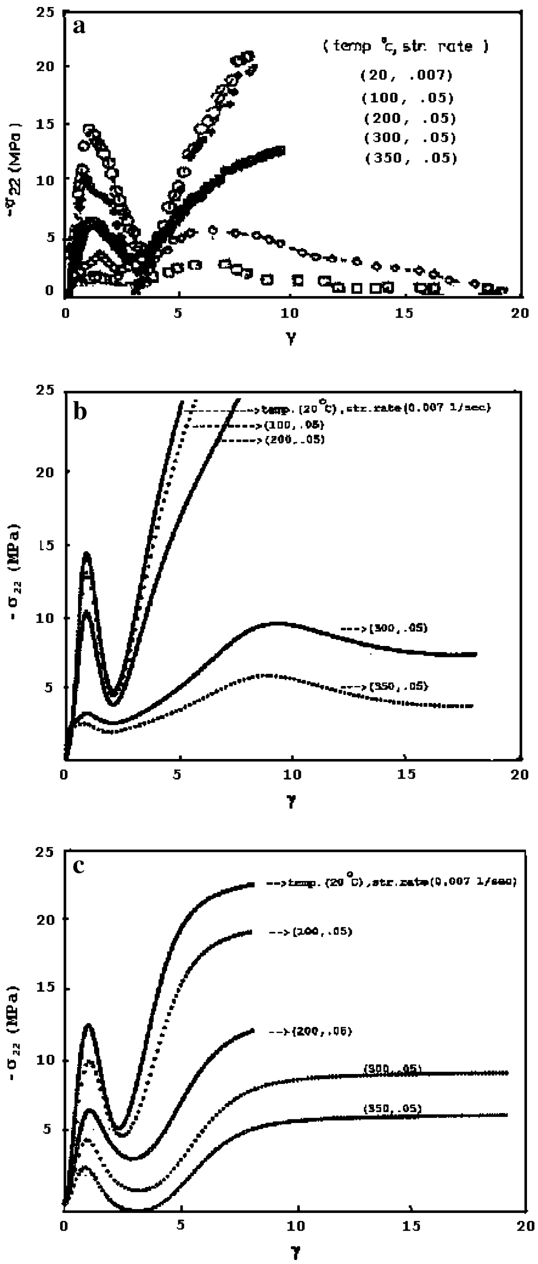

Fig. 4 (**a**) Experimental axial stress–strain curves for Al at different temperatures after Montheillet et al. [13]. (**b**) Simulation of the experimental results for Al at different temperatures with the proposed model. (**c**) Simulation of the experimental results for Al at different temperatures after Cho and Dafalias [12]

unity. As is released from such a comparison the two descriptions are coincident at initial deformations.

Conclusions

One of the most widely accepted constitutive plasticity theories capable of describing all the complementary effects of yielding, such as rate effect, and each one of the hardening features, namely isotropic, kinematic, distortional and orientational hardening is the one developed by Dafalias and Cho [12], that allows an independent chose of plastic spin expressions, followed by a specific flow rule for each internal variable.

Table 2 Model constants for the simulation of the experimental data of Fig. 4

Model constants	Al (20 °C) (0.007 s^{-1})	Al (100 °C) (0.05 s^{-1})	Al (200 °C) (0.05 s^{-1})	Al (300 °C) (0.05 s^{-1})	Al (350 °C) (0.05 s^{-1})
G (MPa)	80	50	25	5	3
ω_{12}	3	3	3	2	2
$\omega_{13} = \omega_{23}$	0	0	0	0	0
C (MPa)	5	10	15	20	20
t_e (s)	100	100	40	20	20

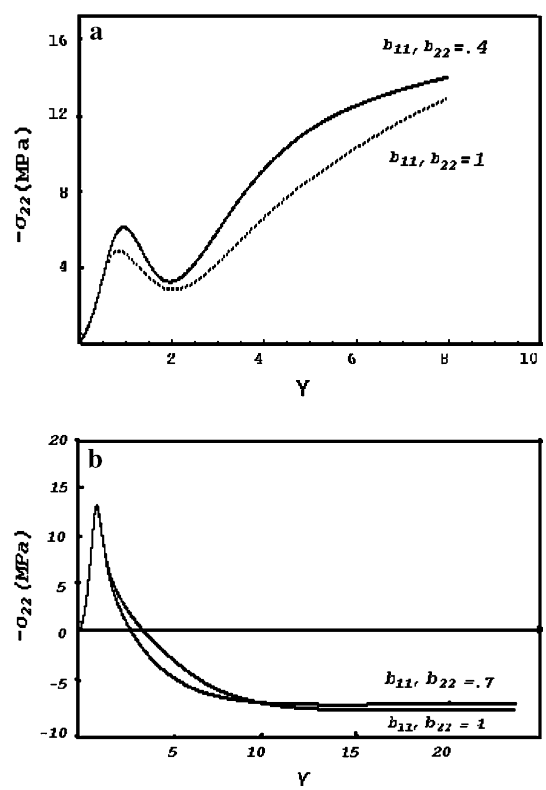

Fig. 5 (**a**) Plots showing the solution of constitutive equations for axial stress–strain curves of Al at 200 °C under loading with 0.05 s^{-1} strain rate. The isotropic behavior has been calculated with material constants $b_{ij} = 1$ ($i,j = 1,2,3$), and the anisotropic response with $b_{11} = b_{22} = 0.4$ ($b_{12} = b_{13} = b_{23} = b_{33} = 1$). (**b**) Plots showing the solution of constitutive equations for axial stress–strain curves of Cu at 300 °C under loading with 0.05 s^{-1} strain rate. The isotropic behavior has been calculated with material constants $b_{ij} = 1$ ($i,j = 1,2,3$), and the anisotropic response with $b_{11} = b_{22} = 0.7$ ($b_{12} = b_{13} = b_{23} = b_{33} = 1$)

In this work a new three-dimensional viscoplastic model based on the plasticity theory developed by Rubin [2, 3] is presented. The proposed constitutive model anticipates the contribution of the main features of plastic behavior, such as yielding, rate effect, isotropic and kinematic hardening, through a new approximation of the constitutive equation with a viscoplastic term, as well as a new consideration of the functional form of the rate of plastic deformation. A high accuracy simulation of torsion experimental data at various rates and temperatures for a variety of materials, as well as the sign inversion of normal stress has been postulated. Moreover, the required number of parameters is minimized in respect to other theories, while all parameters used and their magnitudes have a physical significance.

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